CHAPTER 1
Puzzling about (mathematical) thinking

> One… fact must astonish us or would rather astonish us if we were not so much accustomed to it: How does it happen that there are people who do not understand mathematics? If this science invokes only the rules of logic, those accepted by well-formed minds, how does it happen that there are so many people who are impervious to it?
>
> Henri Poincaré[1]

Full of puzzles, mathematics is a puzzle in itself. Anybody who knows anything about it is likely to have questions to ask. Most of us marvel on how abstract mathematics is and wonder how one can come to grips with anything as complex and as detached from anything tangible as this. The concern of those who do manage the complexity, as did the French mathematician and philosopher of science Henri Poincaré is just the opposite: The fortunate few who "speak mathematics" as effortlessly as they converse in their mother-tongue have a hard time trying to understand other people's difficulty. From a certain point in our lives, it seems, mathematical understanding becomes the "all or nothing" phenomenon – either you have it, or you don't, and being in any of these two camps appears so natural that you are unable to imagine what it means to be in the other.

But the bafflement with regard to mathematics goes further than that. Literature about human thinking is teeming with resilient mathematics-related puzzles. Some of these puzzles are well-known and have been fueling vocal debates for a long time now; some others are still waiting for broader attention. Let me instantiate both types of quandaries with a number of examples. Each of the five stories that follow begins with a brief description of a well-documented controversy and continues with additional teasing questions that must occur to us the moment we manage to see a familiar situation in unfamiliar light. No solutions will be proposed at this time, and when the chapter ends, some readers may feel left midair, and rather annoyingly so. May I thus beg for your patience: Grappling with the conundrums that follow is going to take this whole book. In this chapter, my aim is to present the maladies of the present research on thinking and prepare the ground for diagnosing their sources. The attempt to follow with a cure will be made in the remaining chapters. I do hope that the long journey toward a better understanding of thinking will be not any less rewarding than the prizes that wait at its end.

1. The quandary of ***number***

Puzzling phenomena related to mathematics can be observed already in the earliest stages in child's development. Some of the best known and most discussed of such phenomena were first noticed and documented by the Swiss psychologist Jean Piaget.[2] To put it in Piaget's own language, young children do not *conserve number*, that is, they are not aware of the fact that mere spatial rearrangements do not change cardinality of sets of objects (or, to put it simpler, as long as nothing was added or taken away, the counting process, if repeated, always ends with the same number-word).

A child's awareness of the conservation of number is tested with the help of specially designed tasks. In one of such tasks the child is shown two numerically equivalent sets of counters arranged in parallel rows of equal length and density. The one-to-one correspondence of the counters is thus readily visible when the child is asked "Which of the rows has more marbles?" In this situation, even young interviewees are reported to give the expected answer. One of the rows is then stretched so as to become longer without becoming more numerous. The child is then asked the same question again. On the basis of their performance, most of 4 and 5-year old children are believed to be at "pre-conservation" stage: When requested to compare the rows of the unequal length, even those of them who previously answered that "no row has more" now point to the one that has been stretched. This phenomenon appears particularly surprising in the view of the fact that by the age of 4 the majority of children have already mastered the art of counting up to 10 or 20.[3] Why is it that children who can count properly do not turn to counting when presented with the question "Which of the two rows has more marbles?" "They do not yet conserve number" is a traditional Piagetian answer. Piaget's perplexing finding, as well as his diagnoses, led to a long series of additional studies in which 4 and 5 year old children were presented with tasks best to be solved with the help of counting, such as set comparison or construction of numerically equivalent sets. All these studies confirmed at least one of Piaget's observations: Although skillful in counting, children tend to perform certain tasks with non-numerical methods, which more often than not lead them to "nonstandard" results.

Over the last several decades these phenomena and their Piagetian interpretation generated much discussion.[4] For example, Margret Donaldson and James McGarrigle[5] speculated that children may have at least two good reasons to modify their answers following the change in the arrangement of sets, with none of these reasons translating into the young learners' "inability to conserve number." First, it seemed plausible that rather than relating the words *has more* to the cardinality of sets, the children attend to the immediately visible properties of the rows, such as length. Second, according to the rules of the learning-teaching game widely practiced both in schools and at children's home, the very reiteration of the question may be interpreted by the young interviewees as a prompt for a change in the answer [6].

In the attempt to have a closer look at this phenomenon, my colleague Irit Lavi and I have launched an *Incipient Numerical Thinking Study,*[7] in which we intended to check the numerical skills of Irit's four-year old daughter, Roni, and of Roni's seven-months-older friend Eynat. Our intention was to conduct an experiment not unlike those described above: We would ask the girls to compare sets of counters. Although our study led to findings not unlike those obtained by Piaget and his followers, it also became a source of new, previously unreported quandaries. One vignette from this study suffices to exemplify certain striking, previously unreported, aspects of the children's performances. Episode 1, presented below, is the beginning of the first, 20-minute-long conversation between the two girls and Roni's mother. The event took place in Roni's house. Two sets of marbles were presented to the girls in identical closed boxes, with the marbles themselves invisible through the opaque walls.[8]

Fig. 1:
Roni, 4 (left) and
Eynat, 4 (7)

Episode 1.1: Comparing sets of marbles

Speaker	What is said	What is done
1. Mother	I brought you two boxes. *Do you know what is there in the boxes?*	*Puts two identical closed opaque boxes, A and B, on the carpet, next to the girls.*
2. Roni	Yes, marbles.	
3a. Mother	Right, there are marbles in the boxes.	
3b. Mother	I want you to tell me in which box there are more marbles.	*While saying this, points to box A close to Eynat, then to box B.*
3c. Eynat		*Points to box A, which is closer to her.*
3d. Roni		*Points to box A*
4. Mother	In this one? How do you know?	*Points to box A*
5. Roni	Because this is the biggest than	*While saying "than this one" points to*

	this one. It is the most.	*box B, which is closer to her*
6. Mother	Eynat, how do you know?	
7. Eynat	Because… cause it is more huge than that.	*Repeats Roni's pointing movement to box B when saying "than that"*
8. Mother	Yes? This is more huge than that? Roni, what do you say?	*Repeats Roni's pointing movement to box B when saying "than that"*
9. Roni	That this is also more huge than this.	*Repeats Roni's pointing movement to box B when saying "than that"*
……	**……**	*……*
10a. Mother	Do you want to open and discover? Let's open and see what there is inside. Take a look now.	
10b. Roni		*Abruptly grabs box A, which is nearer to Eyant and which was previously chosen as the one with more marbles.*
11. Roni	1.. 1.. 1.. 2, 3, 4, 5, 6, 7, 8.	*Opens box A and counts properly.*
12. Eynat	1, 2, 3, 4, 5, 6.	*Opens box B and counts properly.*
13. Mother	So, what do you say?	
14. Roni	6.	
15. Mother	Six what? You say 6 what? What does it mean "six"? Explain.	
16. Roni	That this is too many.	
17. Mother	That this is too much? Eynat, what do you say?	
18. Eynat	That this too is a little.	
19. Mother	That it seems to you a little? Where do you think there are more marbles?	
20. Roni	I think here.	*Points on the box, which is now close to her (and in which she found 8 marbles)*
21. Mother	You think here? And what do you think, Eynat?	
22. Eynat	Also here.	

As predicted by the mother, the girls have shown full mastery of counting. In spite of this, they did not bother to count the marbles or even to open the boxes when asked to compare the invisible contents. Their immediate response was the choice of one of the closed boxes ([3c], [3d]). Not only did they make this instant move and agreed in their decision, but they were also perfectly able to "justify" their action in the way that could have appeared adequate if not for the fact that the girls had no grounds for the comparative claims, such as "this is the biggest than this one" ([5]), "It is the most" ([7]) and "it is more huge than that" ([9]). If the startled mother had hoped that her interrogation about the reasons for the choices ([4], [6], [8]) would stimulate opening the boxes and counting the marbles, she was quickly disillusioned: Nothing less than the explicit request to open the boxes ([10a]) seemed to help.

By now, we are so familiar with the fact that "children who know how to count may not use counting to compare sets with respect to number,"[9] that the episode may fail to surprise us, at least at the first reading. And yet, knowing what children usually *do not* do is not enough to account for what they *actually do*. Our young interviewees' insistence on deciding which box "has more marbles" without performing any explorations is a puzzle, one that has not been noted or accounted for in the previous studies. Unlike in conservation tasks, Roni and Eynat made their claims about the inequality without actually seeing the sets, so we cannot ascribe their choices to any visible differences between the objects of comparison. Neither can the children's surprising decision be seen as motivated by the rule "Repeated question means 'Change your answer!'": The girls chose one of the indistinguishable boxes already the first time round, before the parents had a chance to reiterate their request. Well, they were playing a guessing game, somebody may say. This would mean that the children knew they would have to verify their guess by counting the contents of the two boxes. However, neither of them seemed inclined to actually perform such a verifying procedure, and when they eventually did, there was no sign they were concerned with the question of whether the present answer matches the former direct choice. Moreover, the hypothesis of a guessing game, even if confirmed, still leaves many questions unanswered: Why were the girls in such perfect accord about their choices even though these choices seemed arbitrary? What was it that evidently made the chosen box so highly desirable? (note that each of the girls wanted this box for herself, see e.g., [10d]). Why after making the seemingly inexplicable decisions were the children able to answer the request for justification? On what grounds did they claim that what they chose is "the biggest" or "more huge"? Many different conjectures may be formulated in an attempt to respond to all these queries, but it seems that a real breakthrough in our understanding of children's number-related actions is unlikely to occur unless there is some fundamental change in our thinking about numerical thinking.

It seems that in order to come to grips with these and similar phenomena, one needs to go beyond the Piagetian frame of mind. Indeed, if there is little in the past research to help us account for what we saw in this study, it is probably because theory-guided researchers used to attend to nothing except for those actions of their interviewees which they have classified in advance as relevant to their study, and for the Piagetian investigator, the conversation that preceded opening of the boxes would be dismissed as a mere 'noise'. The analysis of the remaining half of the event might even lead her to the claim that Roni and Eynat had a satisfactory command over numerical comparisons, although this is not the vision that emerges when the second part of the episode is analyzed in the context of the first.

2. The quandary of *abstraction* (and *transfer*)

The most common explanation of the wide-spread failure in the more advanced school-type mathematics is its highly *abstract* character. Abstracting, the specialty of scientists at large and of mathematicians in particular, has always been a highly valued activity, appreciated for its power to generate useful generalizations. It has been believed that if people engage in abstract thinking in spite of its difficulty, it is because of the natural tendency of the human mind for organizing one's experience with the help of unifying patterns and structures. It may be thus surprising that the notion of abstraction has been getting bad press lately. True, the troubles did not really start today. The idea of abstraction boggled the minds of philosophers and of psychologists ever since the birth of their disciplines, and critical voices, pointing to abstraction-engendered conceptual dilemmas, could be heard for centuries. And yet, never before was it suggested, as it is now, that the term *abstraction* be simply removed from the discourse on learning.[10]

To get a flavor of the phenomena that shook researchers' confidence in the human propensity for abstracting, let us look at the brief episode that comes from the study of Brazilian street vendors conducted by Teresa Nunes, Annalucia Schliemann and David Carraher.[11] The 12 year-old child, M, selling coconuts at the price of 35 cruzeiros per unit, is approached by a customer.

CUSTOMER:	I'm going to take four coconuts. How much is that?
M, THE CHILD:	There will be one hundred five, plus thirty, that's one thirty-five… one coconut is thirty-five… that is… one forty!

Some time later, the child is asked to perform the numerical calculation $4 \cdot 35$ without any direct reference to coconuts or money.

CHILD:	Four times five is twenty, carry the two; two plus three is five, times four is twenty. [Answer written: 200]

The new result, so dramatically different from the former, may seem puzzling to anybody who knows a thing or two about mathematics. To put it in the researchers' own words, "How is it possible that children capable of solving a computational problem in the natural situation will fail to solve the same problem when it is taken out of context?"[12] Solving "the same problem" in different situations means being able to view the two situations as, in a sense, the same, or at least as sufficiently similar to allow for application of the same algorithm. Being able to notice the sameness (or just similarity) is the gist of abstracting, and the capacity for abstracting is said to be part and parcel of the human ability to "transfer knowledge" – to recycle old problem-solving procedures in new situations. What puzzled the implementers of the Brazilian study was the fact that this latter ability seemed to be absent in M, as well as in practically all the other young street vendors whom they interviewed.

One may try to account for these findings simply by saying that the main reason for the disparity between the Brazilian children performances in the street and in school-like situations was their insufficient schooling. M's inability to cope with the abstract task is understandable in the view of his almost complete lack of school learning. And yet, the question remains why it did not occur to the child to use in the school-like situation the very same algorithm that made him so successful in the street. This query becomes even more nagging in the view of the results of other cross-cultural and cross-situational studies, most of which indicated that people who are extremely skillful in solving everyday mathematical problems may have considerable difficulty with learning abstract equivalents of the real-life procedures. Consider, for example, the findings of the study conducted by Michael Cole and his colleagues in the early 1960s in Liberia. Although the Kpelle people, whom the researchers observed, have shown the great agility in operating on quantities of rice and in money transactions, they seemed almost impervious to school mathematics. "Teachers complained that when they presented a problem like 2+6=? as an example in the classroom and then asked 3+5=? on a test, students were likely to protest that the test was unfair because it contained material not covered in the lesson".[13] Even in retrospect, Cole cannot overcome his bafflement:

> The question aroused by these observations remains with me to this day. Judged by the way they do puzzles or study for mathematics in school, the Kpelle appeared dumb; judged by their behavior in markets, taxis, and many other settings, they appeared smart (at least, smarter than one American visitor). How could people be so dumb and so smart at the same time.[14]

These findings are not unlike the results of many other cross-cultural and cross-situational studies, notably those on dairy warehouse workers,[15] on American shoppers and weight-watchers,[16] and on Nepal shopkeepers.[17] In our own study, we have seen that a child may have difficulty putting together everyday and abstract mathematical procedures even if she has a reasonable knowledge of school mathematics. Consider, for example, two excerpts from an interview with a 12 year-old 7th-grader,[18] whom I shall call Ron. In the first part of the conversation, the child was playing the part of a shop attendant and the interviewer presented herself as a client. The products were represented by cards featuring their names along with their authentic prices. The "vendor" and the "buyer" had a certain amount of real coins and banknotes at their disposal. In the episode that follows, the shopkeeper is calculating the sum to be paid by the buyer, who is asking for 3 cans of tuna fish for 4.99 shekel each and for two bottles of mineral water, for 1.10 shekel each.[19]

Episode 1.2: Utilizing rounding procedure and distributivity

67.	Interviewer:	Three cans of tuna and two bottles of water.	The necessary operation: $3 \cdot 4.99_{IS} + 2 \cdot 1.10_{IS}$
68.	Ron:	[….][20] Two twenty [2.20] […………………] Can I round the sums up?	
69.	Interviewer:	Just tell me how much I am supposed to pay.	
70.	Ron:	[……………….] I think it is 17 [IS] and 17 agoras. [17.17]	
		[…..] Perhaps not.	
		Let me see. I calculated this as 5…. It makes 15, because I multiplied by 3. Minus three agoras from the 99, and it makes 14 and 97 agoras. I added 97 agoras and the 20 agoras of the water and this means I have to add shekel and 17 agoras. It is already 15 and 17 agoras. I added the 2 shekels of the water and this made 17 shekels and 17 agoras.	Performs: $3 \cdot 5_{IS} = 15_{IS}$ $15_{IS} - 3_{ag} = 14.97_{IS}$ $97_{ag} + 20_{ag} = 1.17_{IS}$ $14_{IS} + 1.17_{IS} = 15.17_{IS}$ $15.17_{IS} + 2_{IS} = 17.17_{IS}$

The shopping tasks were followed by purely numerical assignments, one of which was the multiplication 24 · 9. Ron performed the operation based on distributive property and without using the rounding procedure, which might have given the result quicker.

192.	Ron:	[reads] 24 times 9 […...]	24 · 9
		20 times 9 is 180. 9 times 4 is 36. 80 plus 36 is […..…..] 116. 180 plus 36 is 226.	

In spite of his skillfulness in applying the rounding procedure and in taking advantage of distributivity, which he displayed both here and in the previous task with money, Ron did not have recourse to these methods while trying to perform a more complex calculation, 49 · 16:

196.	Ron:	40 times 16. 40 times 10 is 400.	40 · 10 = 400
		9 times 6 is 54. It's 454.	9 · 6 = 54
			400 + 54 = 454
197.	Interviewer:	Is this reasonable?	
198.	Ron:	Why not?	

The interviewer waited for a few moments and then decided to clue the boy toward the use of the rounding procedure.

199.	Interviewer:	Look again at the expression. Are there other similar numbers you could..	49 · 16
200.	Ron:	What do you mean?	
201.	Interviewer:	49 is like…	

202. Ron:	[…..] 64?		64 was a number obtained in one of the former purely numerical tasks.
203. Interviewer:	Do you remember what we did while shopping?		
204. Ron:	[……..]		looks at the prices written on the product-cards
	When we had 99 agoras?		
205. Interviewer:	Yes. What did we do then?		
206. Ron:	We took 1 agora away… It is 50 times 10 minus [….] I turned 49 into 50 [….] 50 times 10 is 500 [….] and 50 times six is 300. It is 800 [….]; and then… 166? No, 346.		50 · 10 - … 50 · 10 = 500 50 · 6 = 300 500 + 300 = 800 800 - 166? 800 - 346?
207. Interviewer:	You subtract 346?		
208. Ron:	Yes, 346.		

Ron's present difficulty with utilizing the rounding procedure and the distributive property, which clearly contrasts with the facility he demonstrated while applying both of them in the 'real life' situation, may be due to the difference in the numbers involved. However, it may also be a result of the fact that this time, the calculations were performed on the "bare" numbers and not on the familiar notes and coins which evidently mediated – either in their real or only imagined form – the earlier real-life calculation. Not to speak about the possibility that Ron might simply have no reason to associate the paper-and-pencil numerical tasks with the money transactions. After all, numerical tasks are performed in schools to show one's mastery of formal computational procedures, not merely to produce an answer. Whichever the reason, the question is what can be done to overcome this compartmentalization of techniques.

According to many researchers, the bulky findings that indicate the strong dependence of human actions on the situations in which the actions take place seem to undermine the underlying assumption that abstract concepts and procedures, once learned, will readily "transfer" to new situations whenever such possibility offers itself:

> recent investigations of learning.. challenge.. separating of what is learned from how it is learned and used. The activity in which knowledge is developed and deployed, it is now argued, is not separable or ancillary to learning or cognition. Nor is it neutral. Rather, it is an integral part of what is learned. Situations might be said to co-produce knowledge through activity. Learning and cognition, it is now possible to argue, are fundamentally situated.[21]

The resulting criticism of the ideas of abstraction and transfer goes from moderate to radical – from one that focuses on common faults in our understanding of the concept to one that posit its outright untenability. In the radical version, the notion of abstraction, seen as practically inseparable from the issue of the generality of knowledge and from the concept of learning transfer, is being accused of bringing into cognitive research tacit assumptions that are bound to lead this research astray. Thus, for example, the first theorists who proposed to conceptualize learning in terms of participation in certain well-defined practices rather than in terms of "acquiring knowledge" declared that they "challenge… the very meaning of abstraction and/or generalization" and "reject conventional readings of the generalizability and/or abstraction of 'knowledge'."[22] In the moderate version, the proposal is not so much to abandon the idea of abstraction as to be more aware of the hazards of its careless conceptualization and of its perfunctory applications. Referring to the heated controversies between those who oblige abstraction and those who reject it, says James Greeno:[23] "On the issue of abstraction… the disagreement … is about theoretical formulations, rather than being about empirical claim." And further, while

> abstract representations can facilitate learning when students share the interpretive conventions that are intended in their use…., [a]bstract instruction can also be ineffective regarding some important purposes if what is taught in the classroom does not communicate important meanings and significance of symbolic expressions and procedures.

Whether phenomena such as those described above should be taken as showing the inherent, insurmountable situatedness of learning remains a moot point. The discussion on the nature and place of abstraction in human thinking is going on and on and does not show any signs of approaching definite conclusions.[24] Whatever the interpretation and conclusions drawn from the cross-cultural and cross-situational studies, however, one thing is certain: These studies' findings face us with the dilemma. On the other hand, we seem to have good reasons to doubt the effectiveness of what Greeno calls "abstract learning," well exemplified by the type of learning that takes place in mathematics classrooms; on the other hand, even if often disappointing in its immediate results, this type of learning still seems to be the quickest path toward useful reorganization of practices that constitute our lives.[25] Indeed, neither the human civilization, nor our everyday activities would have developed the way they did if not for our capacity for abstracting and generalizing.[26]

3. The quandary of *misconceptions*

Some difficulties with mathematics are wide-spread and well-known to every teacher. In spite of their commonness, many of them are a constant source of wondering and bewilderment. Among the most intriguing phenomena commonplace in a mathematical classroom are those that came to be known as *misconceptions*. We are said to be witnessing a misconception whenever a student is using a certain concept, say function, in a way which, although systematic and invariant across contexts, differs from how this concept is used by experts. Researchers describe this phenomenon as showing that children, in the process of learning mathematics, tend to "*create their own meanings*—meanings that are not appropriate at all."[27] The words "not appropriate" refer not so much to the inner coherence of student's thinking as to possible disparities between students' conceptions and the generally accepted versions of the same ideas. Thus, for example, studies have repeatedly shown that the overwhelming majority of high school students tend to believe that any function must have an underlying algorithm. This conviction persists in spite of the fact that the definition, which most of the learners can repeat without difficulty, does not require any kind of behavioral 'regularity'.[28] Similarly, young children are known to believe that the operation of multiplication must increase the multiplied number, while division must make it smaller.[29] A child's idiosyncratic notions tend to be consistent one with another and are sometimes very difficult to change. All this has been widely documented in literature.[30]

Although today our knowledge about the ways in which children think about numbers, functions, proofs and other mathematical ideas is impressively rich, there are many questions that the theory of misconceptions leaves open. For example, one cannot stop puzzling in the face of the fact that the same misconceptions are held by children speaking different languages, learning with different teachers and according to different curricula, and using different textbooks. How is it that the "misconceiving" children agree among themselves about how to disagree with the definition? Such a well-coordinated rebellion against generally accepted rules of words use cannot be dismissed as just accidental "erring".

The phenomenon of misconceptions is known in other domains of knowledge as well, notably in science, but in mathematics, the striking regularity of the "mistaken" ways of thinking is particularly perplexing. Indeed, the fact that many children hold misconceptions about earth can be accounted for by saying that the classroom is not the only, perhaps even not the most influential, source of one's knowledge about earth. A child's own experiences in the world are the primary type of "material" of which her ideas about earth are forged. Since these experiences are similar in different individuals, no wonder that people's "private" misconceptions are similar one to another as well. And yet, this explanation does not seem to hold for mathematical concepts, many of which are unknown to children at the time they begin learning about them in school. Thus, how can one account for the well coordinated 'distortion' of such a notion as, say, function which, when first encountered in school, does not have any obvious "real world" counterpart? One becomes even more bewildered when one notices the strange similarity between children's misconceptions and the early historical versions of the concepts. Thus, for example, the first definitions presented functions simply as formulas.[31] In this case, it was justified to claim that functions express certain algorithmic regularities – a claim that today counts as misconception. How do abstract mathematical concepts created by mathematicians get life of their own and start dictating to their creators what to think? Why do today's children think like the mathematicians of the past?

The trouble with the idea of misconception does not stop here. In addition to its being insufficiently understood, the notion turns out to be of only limited value as an explanatory tool, supposed to help in accounting for what is actually happening when children grapple with mathematical problems. The classroom episode that follows, taken from the *Montreal Algebra Study*, [32] presents one unsuccessful problem-solving attempt that, although it seems to be involving misconceptions, cannot be explained by this fact alone. In the episode, two 12-year-old boys, Ari and Gur, are grappling together with one of a long series of problems supposed to usher them into algebraic thinking and to help them in learning about function. The boys are dealing with the first of the three questions in Figure 1. On the worksheet, function g(x) has been introduced with the help of the partial table of values and the question requires finding the value of g(6), which does not appear in the table. Before proceeding, the reader is advised to take a good look at Ari and Gur's exchange and try to answer some obvious questions: Do the boys' know how to cope with the problem? Do they display satisfactory understanding of the situation? Does the collaboration contribute in any visible way to their learning? If any of the students experiences difficulty, what is the nature of the problem? How could he be helped? What would be an effective way of overcoming – or preventing altogether – the difficulty he is facing?

Fig. 2: Slope episode -- The activity sheet

A function g(x) is partly represented by the table below. Answer the questions in the box:

x	*g(x)*
0	-5
1	0
2	5
3	10
4	15
5	20

(1) What is *g*(6)? _________
(2) What is *g*(10)? _________
(3) The students in grade 7 were asked to write an expression for the function g(x).

Evan wrote $g(x) = 5(x - 1)$
Amy wrote $g(x) = 3(x - 3) + 2(x - 2)$
Stuart wrote $g(x) = 5x - 5$
Who is right? Why?

Episode 1.3: Finding a value of a function

1.	Ari:	Wait, how do we find out the slope again? No, no, no, no. Slope, no, wait, intercept is negative 5. Slope	is trying to get the expression from the table
2.	Gur:	What are you talking about?	
3.	Ari:	I'm talking about this….	points to the -5 in the right column
		It's 5.	moving his eyes to the next row
4.	Gur:	It doesn't matter if it's on (mumble)	
5.	Ari:	5x. Right?	
6.	Gur:	What's that?	
7.	Ari:	It's the formula, so you can figure it out.	
8.	Gur:	Oh. How'd you get that formula?	
9.	Ari:	and you replace the x by 6.	to do the next task: find g(6)
10.	Gur:	Oh. Ok, I…	
11.	Ari:	Look. Cause the, um the slope, is the zero. Ah, no, the intercept is the zero.	
12.	Gur:	Oh, yeah, yeah, yeah. So you got your	
13.	Ari:	And then you see how many is in between each, like from zero to what	"each": A. points to both columns, indicating that you have to check both "from zero to what": he points to the x column
14.	Gur:	And the slope is, so the slope is 1.	the left counterpart of the right-column 0 is 1
15.	Ari:	Hum? No, the slope, see you look at zero,	"zero": he circles the zero in the x column on Gur's sheet
16.	Gur:	Oh *that* zero, ok. So the slope is minus 5	-5 is the f(x) value when x = 0
17.	Ari:	Yeah. And	
18.	Gur:	How are you supposed to get the other ones?	
19.	Ari:	You look how many times it's going down, like we did before. So it's going down by ones. So then it's easy. This is ah.. by fives. See, it's going down by ones, so you just look here	first points to x column ("going down by ones"), then the f(x) column ("by fives"), and again to f(x) column ("look here")
20.	Gur:	Oh. So it's 5	

21.	Ari:	Yeah. 5x plus	
22.	Gur:	Negative 5.	
23.	Ari:	Do you understand?	
24.	Gur:	Negative 5. Yeah, yeah, ok. So what is g 6?	
25.	Ari:	5 times 6 is 30, plus negative 5 is 25. So we <u>did</u> get it right.	
26.	Gur:	No, but it's - in this column there?	"this column": he points to x column
27.	Ari:	Yeah	
28.	Gur:	Oh, <u>then</u> that makes sense. It's 30 What is g 10? ... 40	Writes '30'
29.	Ari:	20, ah 40. No, 45.	
30.	Gur:	No,	
31.	Ari:	45	
32.	Gur:	because 20	
33.	Ari:	10 times 5 is 50, minus	
34.	Gur:	Well, 5 is 20, so 10 must have 40	points to the two entries in the last row
35.	Ari:	times 5	circles the 10 in g(10) on Gur's sheet
36.	Gur:	Oh, we do that thing. Ok, just trying to find it.	
37.	Ari:	Yeah	
38.	Gur:	Cause I was thinking cause 5 is 20,	points again to the last row of the table
39.	Ari:	It's 45. Yeah	
40.	Gur:	(mumble) So it's 45.	

A cursory glance at the transcript suffices to see that Ari proceeds smoothly and effectively, whereas Gur is unable to cope with the task. Moreover, in spite of Ari's apparently adequate algebraic skills, the conversation that accompanies the process of solving does not seem to help Gur.

So far so good: The basic question about the overall effectiveness of the students' problem-solving efforts does not pose any special difficulty. A difficulty begins when we attempt a move beyond this crude evaluation and venture a quest for a deeper insight into the boys' thinking. Let us try, for example, to diagnose the nature of Gur's problem. The first thing to say would be that "Gur does not understand the concept of function" or, more precisely, "He does not understand what the formula and the table are all about, what is their relation, and how they should be used in the present context". Although certainly true, this statement has little explanatory power. What Tolstoy said about unhappiness, seems to be true also about the lack of understanding: Whoever lacks understanding, fails to understand in his or her own way. We do not know much if we cannot say anything specific about the unique nature of Gur's incomprehension.

Rather than asking *whether* students understand, we now ask *how* they understand. It is here that the notion of misconception comes handy. We could say, for example, that Gur's conception of function, unlike his partner's, is still quite faulty. One look at the transcript now, and we identify the familiar nature of the inadequacy: The sequence [28]-[34] shows that Gur holds the ill-conceived idea of proportionality, according to which values of a function should be proportional to the values of the argument.[33] "Misconception of proportionality" is so common that it even made its way to a popular TV sitcom, *Friends*. In one episode, a person tries to prevent an 18-year-old boy from marrying a 44-year-old woman. He says: "She is so much older than you are. And think about the future: when you are 36, she will be 88". "Yeah, I know," says the boy.

The fact that Gur holds the well-known misconception about function, as significant as it surely is, does not seem to satisfy our need for explanation. The misconception-based account leaves us in the dark about many aspects of the above conversation and, more specifically, about the reasons for Gur's choices and responses. The misconception that certainly plays a role in the last part of the exchange does not account for Gur's earlier responses to the notion of formula. These responses seem as unexpected as they are unhelpful. Moreover, although it is obvious that Gur does struggle for understanding, and although the ideas he wishes to understand do not appear to be very complex (indeed, what could be more straightforward than the principle of plugging a number into the formula in order to calculate the value of the function for this number?), all his efforts prove strangely ineffective – they do not seem to make him one step closer to the understanding of the solution that Ari repeatedly tries to explain. It is not easy to decide what kind of action on the part of the more knowledgeable peer could be of some genuine help.

This example seems to reinforce the conclusion drawn from our two former dilemmas: In order to make sense of what people are doing while engaging in mathematical thinking (or in any thinking at all, for that matter), we need not just additional data, but also, and above all, more developed ways of looking, organized into more penetrating theories of thinking and learning.

4. The quandary of ***learning disability***

Within the current tradition, failure in learning is believed to stem from certain inadequacies in one's cognitive processes. Some of these inadequacies, such as those that produce the common misconceptions described above, are regarded as "normal," that is, as natural, almost inevitable, relatively mild perturbations in the otherwise linear growth of knowledge. Some other difficulties are seen as indicating a more serious condition known as learning disability or LD, for short. Historically, this distinction has its roots in the old nature/nurture dichotomy that assumes the possibility of setting apart phenomena originating in biological factors from those that have their roots in environmental influences. Indeed, the decision to distinguish certain cases of unsuccessful learning from all the others stems from the belief that some difficulties indicate a neurologically grounded "cognitive defect."[34] Over time, this approach has proved problematic in several respects, and the resulting research has stumbled upon difficulties.

First, as a result of the proposed distinction, learning difficulties in mathematics have been studied by two different professional communities who do not really communicate with one another. Specialists in LD speak about deficient *cognitive* and *meta-cognitive skills* and insufficient *neurological functioning* as the main characteristics of students with persistent difficulties in mathematics[35]. In contrast, specialists in mathematics education frame mathematics learning difficulties in the neo-Piagetian language of *misconceptions*[36], faulty *mental schemes* or *tacit models*[37], flawed *concept images*[38], and *buggy algorithms*[39]. The lack of a common language between these two communities reduces their chances of engaging in useful exchanges of ideas or building upon each other's research.

Second, the notion of LD, which some researchers consider indispensable in accounting for more extreme cases of mathematical failure, seems inherently problematic. "Many of the difficulties experienced by the LD field emanate from a failure to answer the seeming straightforward question, 'What is a learning disability?,'" admit the authors of a recent article.[40] The reasons for confusion are many. As long as cerebral mechanisms are not directly investigable, descriptions that speak about the existence of neurological faults cannot be truly operative. Aware of this difficulty, LD researchers have been trying to bypass any explicit mention of neurological factors. At present, the most widely accepted definition of LD, proposed by developmental psychologists, refers to children "who possess 'normal' intellectual ability – they are not mentally retarded – but do not seem to profit from sound instruction despite the fact that they are motivated to learn."[41] However, even those who adopt this definition are well aware of its numerous pitfalls. Many of them acknowledge that the distinction between difficulty experienced *despite* instruction and difficulty that develops *because of* instruction is not as straightforward as the definition seems to imply. No wonder, then, that the results attained with diagnostic methods based on this distinction are regarded by many as debatable.[42] Some authors argue that psychologists who "locate [a] child's problem beneath his skin and between his ears"[43] engage in practices that, through their very dynamics, construct rather than merely identify LD. This latter criticism is in tune with more general attacks on the epistemological premises underlying both LD research and the study of misconceptions.[44] I expand on these epistemological issues in the next chapter.

To shed some additional light on the dilemma of learning disability and to present the problem in more concrete terms, let me introduce two 18-year-old high school students, Mira and Talli, who participated in the Learning Difficulty Study[45] (LD study) conducted by Miriam Ben-Yehuda, Ilana Lavie, Liora Linchevski and myself. At the time we met them, Mira and Talli were grade 11 students in a special vocational school for adolescents with long histories of maladjustment, low achievement, and distinct learning difficulties. Due to discontinuous educational histories, both were older than the norm for their class. Mira had prepared herself for a secretarial job and Talli expected to become a hairdresser. Both girls were described by their mathematics teacher as "extremely weak" in arithmetic. While interviewing them, we had ample opportunity to see that, indeed, the state of their arithmetic fell well below what one would expect from an 18 year old. Even simple multiplication of whole numbers seemed to exceed their computational capacities.

Mira, when asked to tell us the history of her mathematics learning, asserted that as a young child she did not experience any difficulty with calculations. She claimed that her difficulties began some time later:

> In the fourth grade, when we started to multiply… I lost the way… I thought it was not for me... I did want to know how to do it… Sometimes I can do things and succeed…. But when I have to think hard, I give up.... The multiplication table… no use in trying to remember. It is so confusing.

The interviewer followed with the question: "How much is 7·16?" When Mira experienced difficulty multiplying 6 by 7, the following exchange took place:

Episode 1.4a: Mira calculates 7·16

Interviewer:	Do you know how much 6 times 7 is?
Mira:	No.
Interviewer:	And if I asked you to figure it out, what would you do?
Mira:	I would use my fingers. Would count seven times.
Interviewer:	Show us.
Mira:	No.
Interviewer:	Please do.
Mira:	No. I do it silently, so that people won't see.

Later, when Talli tackled the same question, "How much is 7·16?" her computational skills did not appear to be much more advanced:

Episode 1.4b: Talli calculates 716

53. Talli:	I'm not good at multiplication table.
56. Interviewer:	[How much is it], 100? 3000? 500? 5? It does not need to be exact.
Talli:	I don't know….

Talli:	I take down the 7, multiply the 6 with the 7 and the 7 with the 1 and get the answer.	
Interviewer:	Do it, please.	
Talli:	6 multiplied by 7 is 36. Okay? I am asking you. [*laughs*].	6·7 = 36?
Talli:	I take it down. [mumbling] 1 multiplied by 7, it gives 7. So it gives 742.	*Writes:* 4 16 · 7 ---- 122

Their common difficulty with multiplication notwithstanding, the girls differed in more than one way. The teacher told us that Talli, in spite of her problems, was a student "with a genuine potential." In contrast, she described Mira as the "weakest student" in her class who clearly did not have "much chance." The teacher also warned us that any effort on our part to perform arithmetic with Mira would be "a waste of time." The teacher's assessment seemed to be in tune with the girls' appearance and demeanor. Mira wore provocative clothes and heavy makeup, and she behaved and spoke like a helpless child. In contrast, Talli's stern look and plain dark clothes gave the impression of a no-nonsense, mature person who knew exactly what she wanted. And yet later, while listening to the girls at length, we became skeptical about the teacher's remarks on their "mathematical potential."

To understand the nature of the conundrum with which we were faced, let us take a look at what was known about Mira and Talli before our investigation began. Our two interviewees' stories, as told by the rich records we found in their school files, may have differed in details but resembled one another in several important respects.

According to these records, in the course of the first 18 years of their lives, both girls experienced more misfortune and suffering than can be found in other people's entire lifespans. Mira, who was the sixth and last child in her family, and whose father stopped working when she was still very young, was subject to sexual assault at the age of 7. The case had no legal follow-up, but the girl received professional assistance and some time later moved to her married brother's house. Talli, the oldest of three siblings, was 8 years old when her mother began a struggle with a terminal illness. Orphaned from her mother 5 years later, she was sent by the deeply religious father to a boarding school, never to return home. She never agreed to see her father again.

The two girls' educational histories were rather discontinuous. Both were frequently moved from one school to another, never spending more than a couple of years in one place, and sometimes having to join classes with children younger than themselves. Under the occasional care of social workers and psychologists, each of our interviewees underwent certain diagnostic examinations at one time or another. In the girls' school files, we found the results of IQ tests, in which they both scored around average, with their IQ performance scores slightly surpassing their verbal scores. In addition, each file contained a number of general evaluations written at different times by psychologists, social workers, and teachers. In these documents, Mira was described as "having normal intellectual ability, with certain emotional impediments and slight learning disabilities." The LD diagnosis was supported with statements about limitations in Mira's "short term and long term memory" and her "difficulties in areas requiring automation." Although Talli was found to have similar difficulties and limitations, in her case the findings did not lead to an explicit claim about LD. Both girls were said to have a "deficiency in acquired knowledge" and to possess "much unrealized potential." In addition, Mira was described as suffering from occasional "attacks of anxiety" and from a "fear of failure" that manifested itself at times of crisis in withdrawal and in "extreme avoidance." She was also said to be "mentally strong and prepared to invest in those areas in which she had genuine interest." Talli was described as "strongly motivated" to learn, but also as occasionally turning to "suppression mechanisms" while trying to overcome anxieties or to cope with her sense of loneliness.

Finally, the two files contained numerous records about the girls' mathematics. Various tests and teachers' assessments invariably pointed to a serious deficiency in both Mira's and Talli's arithmetical skills. This general agreement notwithstanding, the two girls did differ in the quality of their mathematical performance, at least according to their teachers. In vocational tests that she underwent at age of 16, Mira received the lowest possible score. No mention of this kind of test appeared in Talli's file, but a current teacher's assessment described Talli as "strong in comparison to the school average" and as having a "good command over the four basic arithmetical operations." Both in 10th and 11th grade, her final grades in arithmetic were 95%. In contrast, Mira scored only 75% in both cases.

The stories of Mira and Talli left us with many disquieting questions. First, how can one explain the teachers' positive evaluation of Talli's arithmetic skills? This assessment contrasted strongly with what we saw ourselves in the course of the interview. Second, and more importantly, why is it that the girls did not manage to learn the most basic arithmetic even though they were clearly given many opportunities over the years? They were preoccupied with other, more important problems, somebody might say. This is certainly true, and yet, this obvious answer did not seem to delve deeply enough. Indeed, what is the exact nature of the interplay between life hardships and the ability to learn? In this context, what is the status of Mira's LD diagnosis? Was the LD offered as an independent reason for her failure in learning, preexisting her misfortunes, or was it considered as, in a sense, a result of her life adversities? The list of questions remains long. Most of them may probably be summarized as follows: "What is it that made arithmetic so difficult a target for the two girls, and how was this difficulty related to other spheres of their lives?" Dilemmas such as these continue to perplex teachers, remedial specialists, and researchers, whereas the notion of learning disability, rather than help in solving these quandaries, seems to complicate the matters even further.

5. The quandary of *understanding*

One theme common to all the four dilemmas presented above is that of *understanding*. Each of the quandaries could have been formulated as a question about whether, how or why people do or do not understand mathematics, sometimes even under most favorable of circumstances. The interest in the issue of understanding has been pervading psychological, anthropological and educational literature ever since the landmark call for *meaningful learning*, or *learning-with-understanding* which, more than seven decades ago, signaled the end of the behaviorist era and the beginning of the all new direction in the study of human cognition. When W.A. Brownell issued the plea for the "full recognition of the value of children's experiences" and for making "arithmetic less a challenge to pupil's memory and more a challenge to his intelligence,"[46] his words sounded innovative, and even defiant. Eventually, these words helped to lift the behaviorist ban on the inquiry into the 'black box' of mind. Once the permission to look "inside human head" was given, the issue of understanding turned into one of the central topics of research. Cognitive psychology equated understanding with perfecting mental representations and defined learning-with-understanding as one that effectively relates new knowledge to knowledge already possessed. With its roots in Piaget's theory of mental schemes and with its many branches in the quickly developing new science of cognition, this approach had been flourishing for a few decades, spawning a massive flow of research.[47]

In spite of the impressive advances, researchers agree today that pinpointing the exact meaning of the word *understanding* and finding ways to make the principle of learning-with-understanding operative are extremely difficult tasks. The difficulty begins with the elusiveness of the experience that makes us say "I understand": this experience is difficult to achieve and to sustain, and it is even more difficult to capture and to explain. Let me give a personal example. I can clearly remember the event which, for the first time, made me aware of the degree of my ignorance in this respect. I was a beginning teacher and I discovered to my surprise that students who had a good command over systems of linear equations might still be unable to deal with such questions as "For what value of parameter *q* the given system of linear equations has no solution?" I approached the difficulty nonchalantly, confident that the students will be able to overcome the problem in an hour or two. Contrary to my expectations, several days passed before I felt that the class could cope with parameters. But even then the situation was not as good as I hoped for: at the final test only one student managed to produce fully satisfactory solutions to all the problems. In a private conversation with him I remarked, “It seems that you are the only one in this class who really understood the subject.” To my distress, the praise was greeted with an angry response, "I didn't understand anything! I did what I did but I don't know why it worked". I tried to prove him wrong. I presented him with several other problems, one quite unlike the other, and he solved all of them without visible difficulty. I claimed that this kind of questions just cannot be answered by mechanical application of an algorithm. He kept insisting that he "did not understand anything". We ended up frustrated and puzzled. He felt he did not understand parameters, I sensed that I did not understand understanding.

Reflections on my own history helped, but only to some extent. I could remember myself as a graduate mathematics student passing exams, often quite well, but not always having the sense of true understanding. Some time later I was happy to find out that even people who grew up to become well-known mathematicians were not altogether unfamiliar with this kind of experience. For example, Paul Halmos recalls in his "automatography":

> ... I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother me. I could juggle them quite well... [but] I was stumped by the infinitesimal subtlety of epsilonic analysis. I could read analytic proofs, remember them if I made an effort, and reproduce them, sort of, but I didn't really know what was going on.[48]

Halmos was fortunate enough to eventually find out what the 'real knowing' was all about:[49]

> ... one afternoon something happened. I remember standing at the blackboard in Room 213 of the mathematics building talking with Warren Ambrose and suddenly I understood epsilon. I understood what limits were, and all of that stuff that people were drilling in me became clear. I sat down that afternoon with the calculus textbook by Granville, Smith, and Longley. All of that stuff that previously had not made any sense became obvious...

As implicated in this story, what people call 'true' understanding must involve something that goes beyond the operative ability of solving problems and of proving theorems. But although a person may have no difficulty with diagnosing the degree of her understanding, he or she does not find it equally easy to name the criteria according to which such assessment is made. Many articles and books have already been written in which an attempt was made to understand what understanding is all about, but we still seem to be groping in the dark while trying to capture the gist of this fugitive something that makes us feel we had grasped an essence of a concept, a relation or a proof.

Yet another illustration for the elusiveness of the notion of understanding, at large, and of the term *learning-with-understanding*, in particular, comes from the following conversation between a pre-service teacher and Noa, the 7 year old girl:

Episode 1.5: What is the biggest number?

1. Teacher: Can you count to 10?

2. Noa: Yes. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
3. Teacher: Do you know more than ten?
4. Noa: Yes. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10,11, 12, 13, 14, 15, 16, 17, 18, 19, 20.
5. Teacher: What is the biggest number you can think of?
6. Noa: Million.
7. Teacher: What happens when we add one to million?
8. Noa: Million and one.
9. Teacher: Is it bigger than million?
10. Noa: Yes.
11. Teacher: So what is the biggest number?
12. Noa: Two millions.
13. Teacher: And if we add one to two millions?
14. Noa: It's more than two millions.
15. Teacher: So can one arrive at the biggest number?
16. Noa: Yes.
17. Teacher: Let's assume that *googol* is the biggest number. Can we add one to googol?
18. Noa: Yes. There are numbers bigger than googol.
19. Teacher: So what is the biggest number?
20. Noa: There is no such number!
21. Teacher: Why there is no biggest number?
22. Noa: Because there is always a number which is bigger than that?

Clearly, this very brief exchange becomes for Noa an opportunity for learning. The girl begins the dialogue convinced that there is a number that can be called "the biggest" and she ends emphatically stating the opposite: "There is no such number!" The question is whether this learning may be regarded as learning-with-understanding, and whether it is therefore the desirable kind of learning. To answer this question, one has to look at the way in which the learning occurs. The seemingly most natural thing to say if one approaches the task from the traditional perspective is that the teacher leads the girl to realize the contradiction in her conception of number: Noa views the number set as finite, but she also seems aware of the fact that adding one to any number leads to an even bigger number. These two facts, put together, lead to what is called in literature "a cognitive conflict"[50] – a situation supposed to push a person toward revision of her number schema. This is what Noa eventually does. On the face of it, the change occurs as a result of rational considerations, and may thus count as an instance of learning with understanding.

And yet, something seems missing in this explanation. Why is it that Noa stays quite unimpressed by the contradiction the first time she is asked about the number obtained by adding one? Why doesn't she modify her answer when exposed to the discrepancy for the second time? Why is it that when she eventually puts together the two contradicting claims – the claim that adding one is always possible and always leads to a bigger number, on the one hand, and the claim that there is such a thing as *the* biggest number, on the other hand – her conclusion ends with a question mark rather than with a firm assertion (see [22])? Isn't the girl aware of the logical necessity of this conclusion?

In light of the above observations, it is hardly surprising that methods of "meaningful" teaching "are still not well known, and most mathematics teachers probably must rely on a set of intuitions about quantitative thinking that involves both the importance of meaning – however defined – and computation".[51] James Hiebert and Thomas Carpenter echo this concern when saying that promoting learning with understanding "has been like searching for the Holy Grail;" and they add: "There is a persistent belief in the merits of the goal, but designing school learning environments that successfully promote learning with understanding has been difficult."[52] The mild complaint by researchers who belong to the traditional cognitivist school of thought turns into an essential doubt in the mouth of adherents of alternative conceptual frameworks. The difficulty seems so pervasive, they say, one begins wondering whether finding answers to the nagging questions is only a matter of time. Some representatives of new schools of thought go so far as to consider the possibility that the very idea of understanding may be, in fact, theoretically intractable and thus essentially inapplicable either in research or in everyday schooling practice.[53] Like in the previous four cases, one may conclude that nothing less than re-conceptualization may be necessary to make the quandary disappear.

6. Puzzling about thinking in a nutshell

Five persistent, vexing quandaries were presented in this chapter to show that in spite of the long history of thinking about human thinking and large, and of mathematical thinking in particular, those who try to understand this complex phenomenon may well have yet a long way to go. Indeed, the stories just told left us with a long list of unanswered questions. Just to quote a few representative examples:

- Why is it that children who can count without a glitch do not use counting when asked to compare sets of objects? How can we account for what they actually do? More generally, where does numerical thinking begin, how is its incipient version different from our own and how does it become, eventually, just like that of any other adult persons?
- Why is it that even well-educated people do not apply abstract mathematical procedures in situations in which such use could help them with problems they are trying to solve? More generally, why does people's thinking appear so much dependent on particularities of the situations in which it takes place? Are there any teaching strategies that could be used to counteract this situatedness?
- How can one explain the fact that a child who learned a mathematical concept from a teacher or a textbook 'errs' about this concept in a systematic way? How can we account for

the fact that some of these mistakes are shared by great many children all around the world? Even more puzzlingly, how is it that students' "misconceptions" are often very much like those of the scientists or mathematicians who were the first to think about the concepts in question? Most importantly, since the theory of misconceptions, even if perfected, does not seem likely to suffice as a framework for studying learning of mathematics or science, what is it that this theory is missing?

- If the condition known as "learning disability" is supposed to originate in "natural" rather than environmental factors, why does it seem so tightly related to life stories of those who are diagnosed as learning disabled? Which of the two comes first: learning disability or life hardships? Besides, without a direct access to physiological factors, how are we supposed to distinguish between learning disabilities and "normal" learning difficulties?
- Although we do not seem to hesitate while deciding whether we understand something or not, and although we are only too quick to diagnose other people's understanding, we have considerable difficulty trying to articulate our criteria for this kind of judgment. What is it that we do not yet understand about understanding?

These five quandaries, when taken together, lead to the inevitable conclusion: If in spite of the long history of research so many questions about thinking remain unanswered, it may well be that the reason is in our ways of thinking about thinking. Examining this conjecture is the theme of the next chapter.

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[1] Poincaré, 1952, p. 49
[2] Piaget, 1952
[3] This mastery has been described by Rochelle Gelman and her colleagues (see e.g. Gelman & Gallistel 1978) as the ability to observe three principles of counting: the principle of *one-to-one correspondence*, that is, of assigning exactly one number word to each element of the set that is being counted; the principle of *constant order*, that is, of saying the number words always in the same linear arrangement; and the principle of *cardinality*, that is, the awareness of the fact that correct counting of the given set, if repeated, must end with the same number word.
[4] See e.g. Mehler & Bever, 1967, and McGarrigle & Donaldson, 1974
[5] McGarrigle & Donaldson, 1974
[6] Mehan, 1979
[7] This is a longitudinal study, which has been implemented since 2002. Eynat, whom Roni knew since birth, is a daughter of Roni's parents' friends. Both couples are well-educated professionals. The event took place in Roni's house. For a detailed report on the first part of this study see Sfard & Lavie 2005.
[8] The conversation was held in Hebrew. While translating to English, I made an effort to preserve the idiosyncrasies of the children's word use.
[9] Nunes & Bryant, 1994, p. 35
[10] Lave & Wenger, 1991
[11] Nunes, Schliemann & Carraher, 1993, p. 24
[12] *ibid*, p. 23.
[13] Cole 1998, p. 73. Compare Cole, Gay, Glick, & Sharp, 1971; Scribner & Cole, 1981; Scribner, 1997; Lave, 1988; Hoyles & Noss, 2001.
[14] Cole, 1988,p. 74.
[15] Scribner, 1983/1997
[16] Lave, 1988
[17] Beach, 1995
[18] The interview was conducted in Hebrew by Liron Dekel as a part of her masters thesis.
[19] Israeli shekel, IS, is the Israeli monetary unit. 1 shekel is equivalent to 100 agoras.

[20] The dots in square brackets represent pauses, with two dots being equivalent to a break of one second in speech.
[21] Brown et al., 1989, p. 32
[22] Lave & Wenger, 1991, p. 37
[23] Greeno, 1997, p. 13
[24] See e.g. Lave, 1988, Brown et al., 1989, Lave and Wenger, 1991, and then the recent debate in Educational Researcher: Donmoyer, 1996; Anderson et al., 1996; Greeno, 1997; Sfard, 1998; Cobb & Bowers, 1999.
[25] At this point, two disclaimers are in order. First, when speaking about school-type learning (rather than just school learning) I stress that what really counts is the nature of this learning and not the question in what setting it takes place. Second, lest I am misunderstood as claiming that everyday usefulness is the only possible reason for school learning, let me clarify, that if I restrict the present debate to the question of practical impact of school mathematics, it is only because this is the topic around which the recent controversy evolves.
[26] To those who tend to dismiss this last statement because of their objections to the direction taken by our civilization, let me say that my stance is inquisitive, not normative. Whether one is pleased or displeased with the current state of affairs, nobody can deny that human culture is what sets us apart from other species, and that this uniqueness merits researchers' attention. It seems that one thing that is still in the need of a more thorough investigation is the mechanism through which school-type learning leads to lasting re-organization of human activities and to the incessant, consequential growth in their complexity.
[27] Davis, 1988, p. 9.
[28] Malik, 1980; Markovits, Eylon, & Bruckheimer, 1986; Sfard, 1992; Vinner & Dreyfus,1989
[29] Fishbein, 1987, 1989; Fischbein, Deri, Nello, & Marino, 1985; Harel, Behr, Post, and Lesh, 1989
[30] Smith, diSessa, & Rochelle, 1993, Confrey, 1990; see also studies on related ideas, e.g. *concept images* as in Tall & Vinner, 1981; Vinner, 1983 or *tacit models* as in Fischbein, 1989; See also the tightly related, burgeoning research on conceptual change (Vosniadou, 1994; Schnotz, Vosniadou, & Carretero, 1999).
[31] One of the earliest definitions was formulated in 1718 by Jean Bernoulli. It presented function as "a variable quantity composed in any manner whatever of this variable and of constants." It was followed by the one given in 1737 by Euler, which defined function as "analytic expression" (Kline, 1980).
[32] This study, directed by Carolyn Kieran and myself has been implemented in 1992-1994 in one of the Montreal middle schools, situated in an affluent area. The aim of the 30 session long teaching sequence produced for the sake of the study was to introduce the students to algebra while investigating their ways of constructing algebraic concepts and testing certain hypotheses about possible ways of spurring these constructions. The present episode is taken from the 21st meeting. More information on the study, as well as another outlook at the present episode, may be found in Kieran & Sfard (1999), Sfard & Kieran (2001a, 2001b).
[33] The proportionality belief is a variant of the well-known misconception according to which any function should be linear (Markovitz et al., 1986; Vinner & Dreyfus, 1989; Van Dooren, De Bock, Janssens & Verschaffel, 2005)
[34] Kosc, 1974
[35] Chinn, 1996; Garnett, 1992; Goldman, Pellegrino, & Mertz, 1988; Steeves & Tomey, 1998
[36] Smith, diSessa, & Roschelle, 1993
[37] Dreyfus, 1992; Fischbein, 1989; Hershkowitz, 1989
[38] Tall & Vinner, 1981; Vinner, 1991
[39] Brown & Burton, 1978
[40] Kavale & Forness, 1997, p. 3
[41] Ginsburg, 1997, p. 27; cf. Shaywitz, Escobar, Shaywitz, Fletcher, & Makuch, 1992
[42] Ginsburg, 1997; Hoard, Geary, & Hamson, 1999
[43] Mehan, 1996, p. 268
[44] McDermott, 1993; Mehan, 1996; Varenne & McDermott, 1999
[45] The full report on the study can be found in Ben-Yehuda, Lavi, Linchevski, & Sfard 2005.
[46] Brownell, 1935, p. 31.
[47] See e.g. Hiebert and Carpenter, 1992
[48] Halmos 1985, p. 47
[49] Albers & Alexanderson, 1985, p. 123
[50] See e.g. Tall & Schwartzenberger, 1978
[51] Mayer 1983, p. 77
[52] Hiebert & Carpenter, 1992, p. 65
[53] Edwards, 1993; Lerman, 1999